College Name - S-S-College. B.Sc Part-III
Subject - Math
Time — 10 A.M to 12 A.M Topic - Uniform Convergence (Real analysis)
Date — 14-05-2020 By - Shree Nath Sharma

8. State and Prove the Weierstrass Approximation theorem -

Statement — Let f be a Continuous real Function defined on closed interval $[a,b]$ and let $\varepsilon>0$ be given then there exists a polynomial p with real co-efficients such that $|f(x)-p(x)|<\varepsilon$ for all $x\in[a,b]$

P: → STEP I →

First we prove the result for closed interval $[0,1]$

For $x\in[0,1]$,

we have
$$\sum_{k=0}^{n} {}^nC_k (1-x)^{n-k} x^k = (1-x+x)^n = 1 \quad \text{——— (1)}$$

differentiating (1) with respect to x, we get

$$\sum_{k=0}^{n} {}^nC_k \left[-(n-k)(1-x)^{n-k-1} x^k + k(1-x)^{n-k} x^{k-1}\right] = 0$$

$$\sum_{k=0}^{n} {}^nC_k\, x^{k-1} (1-x)^{n-k-1} (k-nx) = 0 \quad \text{——— (2)}$$

Multiplying this by $x(1-x)$, we get

$$\sum_{k=0}^{n} {}^nC_k\, x^k (1-x)^{n-k} (k-nx) = 0 \quad \text{——— (3)}$$

We differentiate (3) with respect to x Considering $x^k(1-x)^{n-k}$ as one of the two factors in applying the product rule. this using differentiation (2) of (1) we get

$$\sum_{k=0}^{n} {}^nC_k \left[x^{k-1}(1-x)^{n-k-1}(k-nx)^2 - nx^k(1-x)^{n-k}\right] = 0$$

i.e $\sum_{k=0}^{n} {}^nC_k x^{k-1} (1-x)^{n-k-1} (k-nx)^2 = n \sum_{k=0}^{n} {}^nC_k x^k (1-x)^{n-k} = n$

Multiplying both sides by $x(1-x)$ and dividing both sides by n^2, we get.

$$\sum {}^nC_k x^k (1-x)^{n-k} \left(x-\frac{k}{n}\right)^2 = \frac{x(1-x)}{n} \quad \text{--- (4)}$$

Now we consider Bernstein polynomial associated with f and defined for each n by

$$B_n(x) = \sum_{k=0}^{n} {}^nC_k x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \quad \text{--- (5)}$$

We shall prove that $\{B_n(x)\}$ converges uniformly to $f(x)$.

using (1) we have

$$f(x) = \sum_{k=0}^{n} {}^nC_k x^k (1-x)^{n-k} f(x)$$

thus $|f(x) - B_n(x)|$

$$= \left| \sum_{k=0}^{n} {}^nC_k x^k (1-x)^{n-k} \left[f(x) - f\left(\frac{k}{n}\right)\right] \right|$$

$$\leq \sum_{k=0}^{n} {}^nC_k x^k (1-x)^{n-k} \left| f(x) - f\left(\frac{k}{n}\right) \right| \quad \text{--- (6)}$$

Since f is continuous on $[0,1]$, it is uniformly continuous on $[0,1]$. So there exists $\delta > 0$ such that

$$\left| x - \frac{k}{n} \right| < \delta \Rightarrow \left| f(x) - f\left(\frac{k}{n}\right) \right| < \varepsilon/2 \quad \text{--- (7)}$$

Now we split the sum on R.H.S of (6) into two parts and denote them by Σ and Σ' where

Σ is the sum of those terms for which $|x - \frac{k}{n}| < \delta$ and

Σ' is the sum of ~~those~~ the remaining terms. (We consider x to be fixed for some moment). Then from (1) and (7) we have

$$\Sigma < \varepsilon/2$$

Since f is continuous it is bounded and so there exists a +ve real no K such that

$$|f(x)| < K \text{ for all } x \in [0,1]$$

Thus we see that

$$\Sigma' \leq 2K\, \Sigma''$$

where $\Sigma'' = \sum {}^nC_k\, x^k (1-x)^{n-k}$ is taken over all k, for which

$$\left|x - \frac{k}{n}\right| \geq \delta,$$

Thus

$$\delta^2\, \Sigma'' \leq \sum {}^nC_k\, x^k (1-x)^{n-k} \left(x - \frac{k}{n}\right)^2 \leq \frac{x(1-x)}{n}$$

i.e $\Sigma'' \leq \frac{x(1-x)}{n\delta^2}$.

The maximum value of $x(1-x)$ on $[0,1]$ is $\frac{1}{4}$. So

$$\Sigma'' \leq \frac{1}{4\delta^2 n}$$

If we choose $n_0 > \frac{K}{\delta^2 \varepsilon}$ then $\Sigma'' \leq \frac{\varepsilon}{4K}$ for all $n \geq n_0$

Thus $\Sigma' < \varepsilon/2$.

Hence $|f(x) - B_n(x)| < \varepsilon$ for $n \geq n_0$ and for all $x \in [0,1]$.

If we assume $p(x) = B_n(x)$ for $n \geq n_0$, we get $|f(x) - p(x)| < \varepsilon$ for all $x \in [0,1]$

Step II - Now we prove the result for any closed $[a,b]$.

If $a=b$, then the result is true for the constant polynomial $p(x) = f(a)$

So we assume that $a<b$

Since $x = f[(b-a)x' + a]$ is a ~~continuous~~ continuous function defined $[0,1]$ ~~to~~ ~~$[a,b]$~~,

~~the function g defined by~~

$g(x)$

So applying result of Step I we get polynomial p' such that

$$|g(x') - p'(x')| < \varepsilon \quad \text{for all } x' \in [0,1]$$

this inequality may also be written as

$$\left| f(x) - p'\frac{x-a}{b-a} \right| < \varepsilon \quad \text{for all } x \in [a,b]$$

If we put $p(x) = p'\dfrac{x-a}{b-a}$ then

$$|f(x) - p(x)| < \varepsilon \quad \text{for all } x \in [a,b]$$

Hence the result

Solved Ex :->

1. Show that the sequence $\{f_n\}$ of functions where

$$f_n(x) = \frac{n}{x+n}$$, is uniformly convergent in $[0,K]$ where K is finite but not uniformly convergent in $[0,\infty]$

Solution :-> The sequence $\{f_n\}$ is point-wise convergent for all $x \geq 0$ and the point-wise limit f is

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n}{x+n} = \lim_{n\to\infty} \frac{n}{n(x/n+1)}$$

$$= \lim_{n\to\infty} \frac{1}{x/n+1} = 1$$

Let $\varepsilon > 0$ be given, we have

$$|f_n(x) - f(x)| = \left|\frac{n}{x+n} - 1\right| = \frac{x}{x+n} < \varepsilon \text{ if}$$

$$n > x\left(\frac{1}{\varepsilon} - 1\right)$$

Let m denote the integer just greater than $x\left(\frac{1}{\varepsilon}-1\right)$. Clearly m increases as x and $m \to \infty$ as $x \to \infty$

Hence we can not find m such that for all $n \geq m$ and for all $x \geq 0$

$$|f_n(x) - f(x)| < \varepsilon$$

Hence the convergence is not uniform in $[0,\infty]$ but if K is finite we can find a m such that

$$m > K\left(\frac{1}{\varepsilon}-1\right) > x\left(\frac{1}{\varepsilon}-1\right)$$

Hence uniformly convergent in ~~[0,K]~~ ~~[0,K]~~ $[0,K]$ where K is some finite.